Cosmological viable Mimetic $f(R)$ and $f(R,T)$ theories via Noether symmetry

D. Momeni

*Eurasian International Center for Theoretical Physics and Department of General &
Theoretical Physics, Eurasian National University,
Astana 010008, Kazakhstan
d.momeni@yahoo.com*

E. Güdekli

Department of Physics, Istanbul University, Istanbul, Turkey

R. Myrzakulov

*Eurasian International Center for Theoretical Physics and Department of General &
Theoretical Physics, Eurasian National University,
Astana 010008, Kazakhstan
rmyrzakulov@gmail.com*

Extended $f(R)$ theories of gravity have been investigated from the symmetry point of
view. We briefly has been investigated Noether symmetry of two types of extended $f(R)$
theories: $f(R,T)$ theory, in which curvature is coupled non minimally to the trace of
energy momentum tensor $T_{\mu\nu}$ and mimetic $f(R)$ gravity, a theory with a scalar field
degree of freedom, but ghost-free and with internal conformal symmetry. In both cases
we write point -like Lagrangian for flat Friedmann-Lemaitre-Robertson-Walker (FLRW)
cosmological background in the presence of ordinary matter. We have been shown that
some classes of models existed with Noether symmetry in these viable extensions of $f(R)$
gravity. As a motivated idea, we have been investigating the stability of the solutions and
the bouncing and ΛCDM models using the Noether symmetries. We have been shown
that in mimetic $f(R)$ gravity bouncing and ΛCDM solutions are possible. Also a class
of solutions with future singularities has been investigated.

Keywords: Modified gravity theories; Cosmology; Noether symmetry; dynamical systems.

1. Introduction

A challenge to the contemporary relativistic cosmology is provided by a set of observational data, indicated on late time acceleration expansion of the whole Universe as well as the initial era, the inflationary epoch [1]-[4]. In the framework of the general relativity (GR), inflationary model, quasi stable de Sitter cannot be realized as a possible physically acceptable model without any extra scalar matter field. It

2   *D. Momeni et al.*

is needed to include some matter fields, like scalar field(s) to resolve it. Another
approach which is totally revolutionary is to replace Einstein gravity by some ex-
tended forms of classical geometrical objects. A reasonable candidate to explain
this situation is modified gravity, in which we replace Einstein-Hilbert action, given
by

$$S_{EH} = \int_{\mathcal{M}} \frac{R}{2\kappa^2} \sqrt{-g} d^4 x + S_{EH}|_{\partial \mathcal{M}} \tag{1}$$

(Here $S_{EH}|_{\partial \mathcal{M}}$ is the boundary term) by another set of geometrical objects like sec-
ond order invariants $R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}, ..$ or functions of Ricci scalar R, Gauss-
Bonnet topological invariant $G = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - 4R^{\mu\nu} R_{\mu\nu} + R^2$ and so on (see [5]-
[14] for reviews). Historically, to be more precise, the simplest potentially reason-
able candidate was $f(R)$ gravity, a theory which it was proposed originally before
recent activities [15]) and later motivated in light of the recent observational data,
as a valid, physically reasonable, ghost-free and stable alternative theory instead of
GR [5]- [14].

Another type of modified gravity is the one in which geometry has been coupled
to the matter fields non- minimally (see [16] for An updated review of such models).
Different types of non-minimally coupled models have been proposed like $f(R, \mathcal{L}_m)$
where $\mathcal{L}_m$ stands for matter Lagrangian and $f(R, T)$ [17],where T is the trace of
the energy momentum tensor of matter fields,which is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} L_{\mathrm{m}} \right)}{\delta g^{\mu\nu}}, \tag{2}$$

Or after a simple checking, it can be rewritten as the following[a]:

$$T_{\mu\nu} = g_{\mu\nu} L_{\mathrm{m}} - 2 \frac{\partial L_{\mathrm{m}}}{\partial g^{\mu\nu}}. \tag{3}$$

In this theory, $T = T_\mu^\mu$. Because of simplicity and beauty form, this theory attracted
several activities in literature [18]-[33]. Symmetry is an important issue to be ad-
dressed in any physical system under study. There are two classes of symmetries:
global symmetries , in which the physical system(dynamical system) respects some
types of transformations, which are defined by functions of coordinates. Another
is local, in which the conservation law gives us a "local" conserved quantity like
charge.

As we mentioned it before, $f(R)$ gravity is a ghost free, and conformally equiva-
lent to the scalar field theory in Einstein frame. There are several interesting features
in this theory to be useful for late time acceleration, dark energy and dark matter
halo problems. Also, inflation can be realized successfully using this simple and ef-
fective modified theory of gravity. Recently to resolve dark matter problem, a new
type of modified gravities has been proposed as titled mimetic model and modified

---

[a]This common definition of energy-momentum tensor can not be used to fix the form of the
Lagrangian. For example, of perfect fluid there is no way to read Lagrangian from the form of $T_{\mu\nu}$

versions of it [34]-[42]. Basic hidden idea behind this new modification of gravity is to propose a conformally invariant, scalar theory of gravity in which scalar degree of freedom does not cause any problem with ghosts. The way is to reparametrize metric as a conformal transformation of an auxiliary metric ,"unphysical" metric. The point is, the scalar field appeared as conformal function and plays the role of an internal degree of freedom. This scalar field is not ghost and it is constructed to have unit norm $\partial_\mu\phi\partial^\mu\phi = -1$. This norm is defined on physical metric. It is remembering for us the role of velocity of a test particle with unit norm in the comoving frame of particle coordinates. An interesting feature is if we write FLRW cosmological equations, an extra term proportional to a^{-3}, appeared. It mimics dark matter. So, it was adequate to name it as "mimetic dark matter" or briefly mimetic gravity. To unify $f(R)$ gravity with this very interesting mimetic theory, it has been proposed mimetic $f(R)$ gravity [43] as a new class of modified gravities with the same inspiration as mimetic theory. Because of its complexity and more physical solutions, this new mimetic $f(R)$ deserves further physical investigations. Very recently the dynamical behavior of mimetic f (R) has been investigated [44]. Our aim here is to address Noether symmetry issue of such mimetic models. In literature Noether symmetry has been investigated for different types of modified gravity like $f(R)$, $f(T)$, Galileons and so on [46]-[60]. Our aim in this paper is to explore Noether symmetry for the mentioned above to modify gravities: mimetic $f(R)$ and $f(R,T)$.

The present paper is organized as follows: formal framework for $f(R,T)$ theory of gravity is presented in Section 2. The formalism of f (R) mimetic theory is motivated in Section 3. The Noether symmetry approach is well understood briefly in Section 4. Noether symmetry is applied to $f(R,T)$ in Section 5 and for $f(R)$ mimetic model in Section 6. We summarize in the last section.

We adopted a "God" given system of units $G = c = 1$, where the gravitational coupling constant is given by $\kappa^2 = 8\pi$.

2. Formal framework of $f(R, T)$ gravity

Let us to start by the following simple extension of $f(R)$ gravity in four dimensional spacetime:

$$S = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}\,d^4x + \int L_{\mathrm{m}}\sqrt{-g}\,d^4x\,, \tag{4}$$

In the action, R stands for the Ricci scalar of the Riemannian space-time, T is trace of the energy, momentum tensor of matter Lagrangian L_{m}, is defined by the simple expression $T = g^{\mu\nu}T_{\mu\nu}$. Trace is an important quantity in quantum theory of inflation and quantum gravity. The dynamical quantity is just metric $g_{\mu\nu}$. So, there is only a single field equation for it. If we calculate the variation of action we obtain:

$$\delta S = \frac{1}{16\pi}\int\left[f_R(R,T)\,\delta R + f_T(R,T)\frac{\delta T}{\delta g^{\mu\nu}}\delta g^{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R,T)\,\delta g^{\mu\nu} + 16\pi\frac{1}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}L_{\mathrm{m}}\right)}{\delta g^{\mu\nu}}\right]\sqrt{-g}d^4x \tag{5}$$

4   *D. Momeni et al.*

In this variation, we define : $f_R(R,T) = \partial f(R,T)/\partial R$ and $f_T(R,T) = \partial f(R,T)/\partial T$, respectively. Since R is the only geometrical quantity in our theory, we recall standard variational expressions for it as the following:

$$\delta R = \delta \left(g^{\mu\nu} R_{\mu\nu} \right) = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \left(\nabla_\lambda \delta \Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta \Gamma^\lambda_{\mu\lambda} \right) , \tag{6}$$

A simple check point is to evaluate it in a local flat coordinate frame, when $\Gamma^\gamma_{\alpha\beta} \equiv 0$. A variation of the connection termination, Γ gives us:

$$\delta \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\nabla_\mu \delta g_{\nu\alpha} + \nabla_\nu \delta g_{\alpha\mu} - \nabla_\alpha \delta g_{\mu\nu} \right) , \tag{7}$$

Consequently, we can write the final simplified form of variation of R as the following:

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g_{\mu\nu} \Box \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu} . \tag{8}$$

By plugging these expressions in the total variation of action, δS, we find:

$$\delta S = \frac{1}{16\pi} \int \left[f_R(R,T) R_{\mu\nu} \delta g^{\mu\nu} + f_R(R,T) g_{\mu\nu} \Box \delta g^{\mu\nu} - f_R(R,T) \nabla_\mu \nabla_\nu \delta g^{\mu\nu} \right.$$
$$\left. + f_T(R,T) \frac{\delta \left(g^{\alpha\beta} T_{\alpha\beta} \right)}{\delta g^{\mu\nu}} \delta g^{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R,T) \delta g^{\mu\nu} + 16\pi \frac{1}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} L_{\mathrm{m}} \right)}{\delta g^{\mu\nu}} \right] \sqrt{-g} d^4 x \tag{9}$$

To simplify more these functional , we need to redefine an auxiliary tensor field $\Theta_{\mu\nu}$ as a part of the variation of trace δT as the following:

$$\frac{\delta \left(g^{\alpha\beta} T_{\alpha\beta} \right)}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu} , \tag{10}$$

And in a similar form as we define T, we are able to define trace of $\Theta_{\mu\nu}$ as the following:

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} . \tag{11}$$

Using these simplifications, we write the following form of the equation of motion of $f(R,T)$:

$$f_R(R,T) R_{\mu\nu} - \frac{1}{2} f(R,T) g_{\mu\nu} + \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_R(R,T) = 8\pi T_{\mu\nu} - f_T(R,T) T_{\mu\nu} - f_T(R,T) \Theta_{\mu\nu} \tag{12}$$

It is remarkable that, when geometry is decoupled from the matter part, in the limit $f(R,T) \equiv f(R)$, we recover equation of motion of $f(R)$ gravity from (12). Trace of (12) gives us the following equation:

$$f_R(R,T) R + 3\Box f_R(R,T) - 2f(R,T) = 8\pi T - f_T(R,T) T - f_T(R,T) \Theta \tag{13}$$

An alternative form of (12) is obtained as the following:

$$f_R(R,T) \left(R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} \right) + \frac{1}{6} f(R,T) g_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu} \right) - f_T(R,T) \left(T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu} \right)$$
$$- f_T(R,T) \left(\Theta_{\mu\nu} - \frac{1}{3} \Theta g_{\mu\nu} \right) + \nabla_\mu \nabla_\nu f_R(R,T) . \tag{14}$$

Conservation of the energy, momentum tensor in this type of modified gravity is checked by computing the covariant divergence of Eq. (12). Using the following geometrical identity we obtain:

$$\nabla^{\mu}\left[f_{R}\left(R,T\right)R_{\mu\nu}-\frac{1}{2}f\left(R,T\right)g_{\mu\nu}+\left(g_{\mu\nu}\Box-\nabla_{\mu}\nabla_{\nu}\right)f_{R}\left(R,T\right)\right]\equiv0\,, \quad (15)$$

So, conservation of energy, momentum tensor, gives us the following vector current term:

$$\nabla^{\mu}T_{\mu\nu}\equiv J_{\nu}=\frac{f_{T}\left(R,T\right)}{8\pi-f_{T}\left(R,T\right)}\left[\left(T_{\mu\nu}+\Theta_{\mu\nu}\right)\nabla^{\mu}\ln f_{T}\left(R,T\right)+\nabla^{\mu}\Theta_{\mu\nu}\right]\,. \quad (16)$$

For a simplicity, we consider the flat FLRW metric in the following form:

$$ds^{2}=-dt^{2}+a^{2}(t)\,\delta_{ij}dx^{i}dx^{j}, \quad (17)$$

Here $a(t)$ stands for the scale factor. If we write down Lagrangian of $f(R,T)$ for this metric and if we assumed that the Universe is filled with matter fields with effective pressure p and energy density ρ, we obtain $T=3p-\rho$. Since this equation is a constraint, and because of $R=6(\frac{\ddot{a}}{a}+(\frac{\dot{a}}{a})^{2})$, we introduce a pair of Lagrange multipliers as the following $\{\lambda,\mu\}$. The point Like Lagrangian after an integration part by part is written as the following (we set $2\kappa^{2}=1$):

$$\mathcal{L}(a,R,T,\dot{a},\dot{R},\dot{T})=a^{3}\Big(f(R,T)-Rf_{R}(R,T)-Tf_{T}(R,T)\Big) \quad (18)$$
$$-6\Big(a\dot{a}^{2}f_{R}(R,T)+a^{2}\dot{a}\dot{R}f_{RR}(R,T)+a^{2}\dot{a}\dot{T}f_{RT}(R,T)\Big)-p(a)a^{3}+a^{3}f_{T}\Big(3p(a)-\rho(a)\Big)$$

Where we suppose that $\mathcal{L}_{m}=-p(a)$. If $f(R,T)=f(R)$, the Lagrangian density is written:

$$\mathcal{L}(a,R,\dot{a},\dot{R})=a^{3}\Big(f(R)-Rf_{R}(R)\Big)-6\Big(a\dot{a}^{2}f_{R}(R)+a^{2}\dot{a}\dot{R}f_{RR}(R)\Big)-p(a)a^{3} \quad (19)$$

This is the form of Lagrangian in $f(R)$ theory. We suppose that (19) is an acceptable assumption for many cosmological applications, like matter dominant era or radiation. The associated equations of motion are given by a set of second order ordinary differential equations, Euler-Lagrange (EL) equations, are given by the following:

$$\frac{d}{dt}\big(\frac{\partial\mathcal{L}}{\partial\dot{q}}\big)-\frac{\partial\mathcal{L}}{\partial q}=0 \quad (20)$$

for $q\equiv\{a,R,T\}$ they are obtained as the foloowing:

$$6\ddot{R}f_{RR}=-3f+3Rf_{R}+3Tf_{T}-6\frac{\dot{a}^{2}}{a^{2}}f_{R}-12\frac{\dot{a}\dot{R}}{a}f_{RR}-12\frac{\dot{a}\dot{T}}{a}f_{RT}+ap'(a)+3\,p\,(a)-9f_{T}p(a) \quad (21)$$

$$+3f_{T}\rho\,(a)-3\,af_{T}p'(a)+af_{T}\rho'(a)\,(a)-6\dot{R}^{2}f_{RRR}-12\dot{R}\dot{T}f_{RRT}-6\dot{T}^{2}f_{RTT}-6f_{RT}\ddot{T}-12\frac{\ddot{a}}{a}f_{R}$$

$$6f_{RR}\frac{\ddot{a}}{a}=f_{RR}\Big(R-6(\frac{\dot{a}}{a})^{2}\Big)+f_{RT}\Big(T-(3p(a)-\rho(a))\Big) \quad (22)$$

$$(R-6\frac{\dot{a}^{2}}{a^{2}}-6\frac{\ddot{a}}{a})f_{RT}+f_{TT}(T-3p(a)+\rho(a))=0. \quad (23)$$

6   *D. Momeni et al.*

To pass the case of $f(R,T) = f(R)$, we obtain:

$$6\ddot{R}f_{RR} = -3f + 3Rf_R - 6\frac{\dot{a}^2}{a^2}f_R - 12\frac{\dot{a}\dot{R}}{a}f_{RR} + ap'(a) + 3\,p\,(a) \qquad (24)$$

$$-6\dot{R}^2 f_{RRR} - 12\frac{\ddot{a}}{a}f_R$$

$$R = 6\Big((\frac{\dot{a}}{a})^2 + \frac{\ddot{a}}{a}\Big). \qquad (25)$$

Our aim in this paper is to investigate Noether symmetry issue of (19). Briefly, we are interested to know how Noether symmetry is able to "fix" mathematical forms of $\{f(R,T), p(a)\}$.

3. formalism of mimetic $f(R)$ gravity

Although $f(R,T)$ provides a reasonable and good extension of $f(R)$ theory, it does not respect conformal symmetry. Also, extra degrees of freedom are possible. So, instabilities due to ghosts probably are happening. To resolve conformal symmetry and to be ghost-free, a model recently proposed as titled Mimetic $F(R)$ gravity [43]. It is inspired from the mimetic theory [34,35,36,37,39,40,42], a model in which dark matter problem is resolved as an integration constant. Also, it is self consistent with conformal symmetry. The basis of any type of mimetic theory is to parameterize of the Riemannian metric tensor $g_{\mu\nu}$ as the following conformally transformed formula [34]

$$g_{\mu\nu} = -\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\hat{g}_{\mu\nu}\,, \qquad (26)$$

Here we introduced a pair of auxiliary objects: the first is an auxiliary metric (unphysical and without dynamics) $\hat{g}_{\mu\nu}$ and the second is a scalar field degree of freedom ϕ which has generally ghost, freedom. It is well known that using an orthogonality of metric, this scalar field satisfies the following constraint equation of motion:

$$g\,(\hat{g}_{\mu\nu}, \phi)^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi = -1\,, \qquad (27)$$

If we know the background metric $g\,(\hat{g}_{\mu\nu}, \phi)^{\mu\nu}$, this equation fixes the form of the scalar field. If we interpret $\partial_\mu\phi$ as the components of a four velocity u_μ, then normalized $u_\mu u^\mu = -1$ implies a possible normalization of the ϕ. This normalization can be understood as the first integral of the equation of motion for ϕ . We emphasize here that the auxiliary metric $\hat{g}_{\mu\nu}$ is an internal object of the space-time manifold Following [43], we write the following action for mimetic $f(R)$ gravity in metric formalism :

$$S = \int d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa^2} + \mathcal{L}_m\right]\,, \qquad (28)$$

Here $\kappa^2 = 8\pi$, R is the Ricci scalar which is computed by the physical metric $g_{\mu\nu}$, and we also include the matter Lagrangian by $\mathcal{L}_m$. What we need is to parametrize

the physical metric according to (26) . We rewrite the action of theory in the following equivalent form:

$$S = \int d^4x \sqrt{-g\left(\hat{g}_{\mu\nu}, \phi\right)} \left[\frac{f\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right)}{2\kappa^2} + \mathcal{L}_m\right] . \tag{29}$$

Thus, we perform variation with respect to auxiliary metric $\hat{g}_{\mu\nu}$, we obtain:

$$\frac{1}{2}g_{\mu\nu}f\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right) - R\left(\hat{g}_{\mu\nu}, \phi\right)_{\mu\nu} f_R\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right)$$

$$+\nabla\left(g\left(\hat{g}_{\mu\nu}, \phi\right)_{\mu\nu}\right)_\mu \nabla\left(g\left(\hat{g}_{\mu\nu}, \phi\right)_{\mu\nu}\right)_\nu f_R\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right)$$

$$-g\left(\hat{g}_{\mu\nu}, \phi\right)_{\mu\nu} \square\left(\hat{g}_{\mu\nu}, \phi\right) f_R\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right) + \kappa^2 T_{\mu\nu}$$

$$+\partial_\mu\phi\partial_\nu\phi\Big[2f\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right) - R\left(\hat{g}_{\mu\nu}, \phi\right) f_R\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right)$$

$$-3\square\left(g\left(\hat{g}_{\mu\nu}, \phi\right)_{\mu\nu}\right) f_R\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right) + \kappa^2 T\Big] = 0\,, \tag{30}$$

As a convention, here f_R means $\partial f(R)/\partial R$, ∇_μ and $\square$ are different derivative operators with respect to $g_{\mu\nu}$. Also by using a similar "dictionary" as we used in $f(R,T)$, we define $T_{\mu\nu}$ as the effective matter, energy-momentum tensor for $\mathcal{L}_m$ is given by (3) . Variation with respect to the scalar field ϕ gives us :

$$\nabla\left(g\left(\hat{g}_{\mu\nu}, \phi\right)_{\mu\nu}\right)^\mu \left\{\partial_\mu\phi\Big[2f\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right) - R\left(\hat{g}_{\mu\nu}, \phi\right) f_R\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right)\right.$$

$$\left. -3\square\left(g\left(\hat{g}_{\mu\nu}, \phi\right)_{\mu\nu}\right) f_R\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right) + \kappa^2 T\Big]\right\} = 0\,, \tag{31}$$

Here like $f(R,T)$, we define trace of the energy-momentum tensor as $T = g\left(\hat{g}_{\mu\nu}, \phi\right)^{\mu\nu} T_{\mu\nu}$. It has been proven that this new theory is conformally invariant and ghost free [43]. So, it is remarkable to consider it as a valid extension of $f(R)$ gravities.

We consider the same FLRW metric as (17). In this case, the constraint equation leads to $\phi = t$ where t is cosmic time. It is not so hard task to write FLRW equations for motion. Actually, because our aim is to investigate Noether symmetry, so what we need is just point like Lagrangian of the mimetic $f(R)$ scenario. To be more generally speaking, we slightly modify the original mimetic $f(R)$ by including a potential term and by introducing a Lagrange multiplier λ as the following:

$$S = \int d^4x \sqrt{-g\left(\hat{g}_{\mu\nu}, \phi\right)} \left[\frac{f\left(R\left(\hat{g}_{\mu\nu}, \phi\right)\right)}{2\kappa^2} - V(\phi) - \lambda(\partial_\mu\phi\partial^\mu\phi + 1) + \mathcal{L}_m\right] . \tag{32}$$

The point like Lagrangian for a fluid with pressure $p(a, \dot{a}, \phi)$ and by taking in to account that $\mathcal{L}_m \equiv p(a, \dot{a}, \phi)$, is written in the following form:

$$\mathcal{L}(a, R, \phi, \dot{a}, \dot{R}, \dot{\phi}) = a^3 f(R) - a^3 R f_R(R) - 6\Big(a\dot{a}^2 f_R(R) + a^2\dot{a}\dot{R}f_{RR}(R)\Big) \tag{33}$$

$$-p(a, \dot{a}, \phi)a^3 - V(\phi)a^3 - \lambda(1 - \dot{\phi}^2)a^3$$

8   *D. Momeni et al.*

Note that in general $\lambda = \lambda(t)$. Equations of motion are written in the following forms:

$$6f_{RR}\ddot{R} = -6\dot{R}^2 f_{RRR} - a\ddot{a}p_{\dot{a}\dot{a}} - \dot{a}ap_{a\dot{a}} - ap_{\dot{a}t} - 3\dot{a}p_{\dot{a}} + ap_a \tag{34}$$

$$-f_R\Big(-3R + 6(\frac{\dot{a}}{a})^2 + 12\frac{\ddot{a}}{a} - 3(f - \lambda + \lambda\dot{\phi}^2 - V(\phi) - p)\Big) - 12\frac{\dot{a}\dot{R}}{a}$$

$$\frac{\ddot{a}}{a} = \frac{1}{6}(R - 6(\frac{\dot{a}}{a})^2) \tag{35}$$

$$\ddot{\phi} + \dot{\phi}(\frac{\dot{\lambda}}{\lambda} + \frac{3\dot{a}}{a}) + \frac{1}{2}\frac{V_\phi}{\lambda} = 0. \tag{36}$$

The second equation is just the standard definition of Ricci scalar for FLRW metric. The third one is reduced to the Klein-Gordon equation in the case of $\lambda \equiv \lambda(t) =$ `Constant`. The first equation becomes familiar as the equation of motion in $f(R)$ gravity, if we set $p = p(a), \lambda = 1, \phi(t) = t, V(\phi) = 0$. Clearly this equation posses de Sitter solution as $R = R_0$, $a(t) = a_0 e^{H_0 t}$.

Because of the importance of this model, we will study fixed points of the associated dynamical system, corresponding to this last case , when $p = p(a), \phi = t$. We applied it before in the context of general relativity [45]. Here we review the basic concepts of an non-autonomous system.

Consider the following differential equation for an non-autonomous dynamical state vector $\vec{x}$:

$$\dot{\vec{x}} = f(t, \vec{x}) \tag{37}$$

$$f : [0, \infty) \times D \longrightarrow \mathbb{R}^n \tag{38}$$

$$D = \{\vec{x} \in \mathbb{R}^n \mid \| \vec{x} \|_2 < 0\} \tag{39}$$

The equilibrium point, or fixed point is located at $x = 0$ if and only if it solves the following algebraic equation for an instant of time,namely t:

$$f(t, 0) = 0, \forall t \geq 0 \tag{40}$$

We define the Jacobian matrix ; $J \equiv [\frac{\partial f}{\partial \vec{x}}]$ must be bounded function of t on a finite domain $\mathcal{D}$ and furthermore it satisfies smoothly the Lipschit'z lemma , as the following:

$$\| f(t, \vec{x}) - f(t, \vec{y}) \| \leq L \| \vec{x} - \vec{y} \|, \| \vec{x} \|_p = (\sum_i^p \mid x_i \mid^p)^{\frac{1}{p}}, 1 \leq p < \infty \tag{41}$$

There is an important theorem about the asymptotic stability of the system in the vicinity of the equilibrium point:

`Theorem` I : It is possible to linearize the system of equations in the vicinity of the fixed point in the following form:

$$\dot{\vec{\delta x}} = A(t)\vec{\delta x} \tag{42}$$

Theorem II: Suppose that $x = 0$ be the fixed point of the system $\dot{x} = f(t,x)$, and it satisfies the following auxiliary conditions:

$$f : [0,\infty) \times D \longrightarrow \mathbb{R}^n$$
$$D = \{\vec{x} \in \mathbb{R}^n \mid \| \vec{x} \|_2 < r\},$$

(43)

It is adequate to define the time dependent function $A(t) = \frac{\partial f(t,x)}{\partial x}\big|_{x=0}$. We say that the system has an `exponential stable equilibrium` point of the linearized system (42) , then this point is the exponential stable equilibrium point of the nonlinear system (37)

In this case the system of equations reduces to the following form:

$$6 f_{RR}\ddot{R} = -6\dot{R}^2 f_{RRR} + a p_a$$

(44)

$$-f_R\Big(-3R + 6(\frac{\dot{a}}{a})^2 + 12\frac{\ddot{a}}{a} - 3\Big(f(R) - \lambda + \lambda\dot{\phi}^2 - V(\phi) - p\Big)\Big) - 12\frac{\dot{a}\dot{R}}{a}$$

$$\frac{\ddot{a}}{a} = \frac{1}{6}(R - 6(\frac{\dot{a}}{a})^2)$$

(45)

$$\frac{\dot{\lambda}}{\lambda} + \frac{3\dot{a}}{a} + \frac{1}{2}\frac{V_\phi}{\lambda} = 0.$$

(46)

The first attempt is done by rewriting the system of equations in terms of a dimensionless "time" coordinate $N = \log a$ and a new set of dimensionless parameters as $x^A = \{h = \frac{H}{H_0}, r = \frac{R}{12H_0^2}, X = r', \zeta = \log\lambda\}$. Equations read as the following non-autonomous system (due to the potential term $V(\phi) = V(N)$:

$$h' = 2(\frac{r}{h} - h)$$

(47)

$$r' = X,$$

(48)

$$X' = -\frac{2X}{h}\Big(\frac{r}{h} - h + \frac{6H_0 f_{rrr}}{f_{rr}}\Big)$$

(49)

$$-\frac{f_r}{144H_0^4 f_{rr}}\Big[-\frac{6r}{h^2} + \frac{4}{h}(\frac{r}{h} - h) - \frac{3}{2H_0^2 h^2}(f - V(N) - p(N))\Big] - \frac{X}{72H_0^2 f_{rr}},$$

$$\zeta' = -3 - \frac{V(N)}{2H_0}\frac{e^{-\zeta}}{h}.$$

(50)

Stationary (fixed) points are located at :

$$f(r_c) = V(N_c) + p(N_c) = 0, \quad X_c = 0, \quad r_c = h_c^2, \quad \frac{V(N_c)}{2H_0}\frac{e^{-\zeta_c}}{h_c} = -3. \quad (51)$$

Because of the critical point is function of N, it's moving when time is running. The Jacobian of the linearized system is given by the following:

$$J = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & 0 & 0 & d_4 \end{bmatrix}$$

(52)

10   *D. Momeni et al.*

Where

$$a_i = \frac{\partial f_1}{\partial q^i}, \quad q^i \equiv \{h, r\}, \tag{53}$$

$$c_i = \frac{\partial f_3}{\partial q^i}, \quad q^i \equiv \{h, r, X\}, \tag{54}$$

$$d_i = \frac{\partial f_4}{\partial q^i}, \quad q^i = \{h, \zeta\}. \tag{55}$$

And:

$$f_1 = 2(\frac{r}{h} - h), \tag{56}$$

$$f_3 = -\frac{2X}{h}\Big(\frac{r}{h} - h + \frac{6H_0 f_{rrr}}{f_{rr}}\Big) \tag{57}$$
$$-\frac{f_r}{144 H_0^4 f_{rr}}\Big[-\frac{6r}{h^2} + \frac{4}{h}(\frac{r}{h} - h) - \frac{3}{2H_0^2 h^2}(f - V(N) - p(N))\Big] - \frac{X}{72 H_0^2 f_{rr}},$$

$$f_4 = -3 - \frac{V(N)}{2H_0}\frac{e^{-\zeta}}{h}. \tag{58}$$

The characteristic equation is given by:

$$\lambda^4 + (-d_4 - c_3 - a_1)\lambda^3 + (d_4 c_3 + d_4 a_1 - c_2 + c_3 a_1)\lambda^2 \tag{59}$$
$$+ (d_4 c_2 - d_4 c_3 a_1 - c_1 a_2 + c_2 a_1)\lambda - d_4(-c_1 a_2 + c_2 a_1) = 0.$$

one eigenvalue is $\lambda_1 = d_4$. Another eigenvalues read as the following:

$$\lambda_2 = \frac{\Delta^2 + 12 c_2 - 4 c_3 a_1 + 4 c_3^2 + 4 a_1^2 + 2 c_3 \Delta + 2 a_1 \Delta}{6\Delta} \tag{60}$$

$$\lambda_3 = -\frac{1}{12}\chi - \frac{c_2}{\chi} + \frac{1}{3}\frac{c_3 a_1}{\chi} - \frac{1}{3}\frac{c_3^2}{\chi} - \frac{1}{3}\frac{a_1^2}{\chi} + \frac{1}{3}c_3 + \frac{1}{3}a_1 \tag{61}$$
$$+\frac{1}{12}i\sqrt{3}\chi - \frac{i\sqrt{3}c_2}{\chi} + \frac{\frac{1}{3}i\sqrt{3}c_3 a_1}{\chi} - \frac{\frac{1}{3}i\sqrt{3}c_3^2}{\chi} - \frac{\frac{1}{3}i\sqrt{3}a_1^2}{\chi}$$

$$\lambda_4 = -\lambda_3^*. \tag{62}$$

In the above expressions :

$$\Delta \equiv 12E + 8 a_1^3 - 12 c_3 a_1^2 + \left(-12 c_3^2 - 72 c_2\right) a_1 + 36 c_2 c_3 + 8 c_3^3 + 108 c_1 a_2, \tag{63}$$

$$E^2 \equiv 54 c_2 c_3 c_1 a_2 - 108 c_2 a_1 c_1 a_2 - 18 c_3^2 a_1 c_1 a_2 - 18 c_3 a_1^2 c_1 a_2 - 12 c_2^3 + 81 c_1^2 a_2^2 - 3 c_2^2 c_3^2 \tag{64}$$
$$+ 24 c_2^2 a_1^2 - 12 c_2 a_1^4 + 6 c_3^3 a_1^3 - 3 c_3^4 a_1^2 - 3 c_3^2 a_1^4 + 12 c_1 a_2 c_3^3 + 12 c_1 a_2 a_1^3 - 24 c_2^2 c_3 a_1$$
$$- 6 c_2 c_3^2 a_1^2 - 6 c_2 c_3^3 a_1 + 24 c_2 c_3 a_1^3$$

$$\chi^3 = 36 c_2 c_3 - 72 c_2 a_1 - 12 c_3^2 a_1 - 12 c_3 a_1^2 + 108 c_1 a_2 + 8 c_3^3 + 8 a_1^3 + 12 \rho, \tag{65}$$

$$\rho^2 = 54 c_2 c_3 c_1 a_2 - 108 c_2 a_1 c_1 a_2 - 18 c_3^2 a_1 c_1 a_2 - 18 c_3 a_1^2 c_1 a_2 + 81 c_1^2 a_2^2 - 3 c_2^2 c_3^2 + 24 c_2^2 a_1^2 \tag{66}$$
$$- 12 c_2 a_1^4 + 6 c_3^3 a_1^3 - 3 c_3^4 a_1^2 - 3 c_3^2 a_1^4 - 12 c_2^3 + 12 c_1 a_2 c_3^3 + 12 c_1 a_2 a_1^3 - 24 c_2^2 c_3 a_1 - 6 c_2 c_3^2 a_1^2$$
$$- 6 c_2 c_3^3 a_1 + 24 c_2 c_3 a_1^3.$$

To have stable solution we must have $d_4 = -3 + \frac{V(N_c)}{2H_0} \frac{e^{-\zeta_c}}{h_c} < 0$. Another eigenvalues λ_i, $2 \le i \le 4$ must satisfy $\Re \lambda_i < 0$. It is remarkable to mention here that the stable manifold is defineb by the de Sitter space-time. So, the model for time intervals $N > N_c$ has stable de Sitter solution.

4. Noether symmetry approach to dynamical systems

Symmetry is an important issue to be addressed in any physical theory. The basic property of a system with a defenite type of symmetry is the existence of an associated conserved quantity under this kind of symmetry. Let us consider a typical dynamical system, is defined by a set of configurations coordinates set q_i. Generally speaking, the dimension of the system is defined as the number of independent coordinates of the system. We assume that the dynamics of the system are defined by the point like Lagrangian is given by: $L \equiv L(q_i, \dot{q}_i; t)$, $1 \le i \le N$. For each coordinate, it is possible to define a "unique" first order conjugate momentum:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i}. \tag{67}$$

Euler-Lagrange equation of motion is given by the following set of N-ordinary second order diffrential equations:

$$\dot{p}_i - \frac{\partial L}{\partial q^i} = 0, \tag{68}$$

What we call it as *Noether Symmetry Approach*[47]-[49] is the existence of a "unique" vector field $\vec{X}$ on tangent space $T\mathcal{Q} \equiv \{q_i, \dot{q}_i\}$): :

$$\vec{X} = \Sigma_{i=1}^{N} \left[\alpha^i(q) \frac{\partial}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial}{\partial \dot{q}^i} \right], \tag{69}$$

If we can find "generators" coefficients $\alpha_i(q_j)$, then we can strictly say that our dynamical system must satisfy the following geometrical constraint (is called as *Lie derivative of the Lagrangian* :

$$L_X \mathcal{L} = 0 \tag{70}$$

where explicitly we have:

$$L_X \mathcal{L} = \vec{X} \mathcal{L} = \Sigma_{i=1}^{N} \left[\alpha^i(q) \frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right]. \tag{71}$$

or equivalently we can write it as the following:

$$\Sigma_{i=1}^{N} \frac{d}{dt} \left(\alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L}. \tag{72}$$

It is an easy task to show that, existence of Noether symmetry implies that the system has the following conserved (local) quantity:

$$\Sigma_0 = \Sigma_{i=1}^{N} \alpha^i p_i \tag{73}$$

The equations are obtained setting to zero the coefficients of the terms $\dot{q}_i{}^m \dot{q}_j^n, 0 \leq m + n \leq N$ in (71). They are several interesting applications of this symmetry approach in different cosmological models in different models [50]-[60]. In our paper we'll apply this approach to $f(R,T)$ and mimetic $f(R)$ theories of gravity.

5. Noether symmetry in $f(R,T)$ gravity

The plan in this section is to study the system of Noether equations for Lagrangian given by (19). If we write down (71) equation for (19) ,

$$
\alpha\frac{\partial\mathcal{L}}{\partial a} + \beta\frac{\partial\mathcal{L}}{\partial R} + \gamma\frac{\partial\mathcal{L}}{\partial T} + \Big(\dot{a}\frac{\partial\alpha}{\partial a} + \dot{R}\frac{\partial\alpha}{\partial R} + \dot{T}\frac{\partial\alpha}{\partial T}\Big)\frac{\partial\mathcal{L}}{\partial\dot{a}} \tag{74}
$$
$$
+\Big(\dot{a}\frac{\partial\beta}{\partial a} + \dot{R}\frac{\partial\beta}{\partial R} + \dot{T}\frac{\partial\beta}{\partial T}\Big)\frac{\partial\mathcal{L}}{\partial\dot{R}} + \Big(\dot{a}\frac{\partial\gamma}{\partial a} + \dot{R}\frac{\partial\gamma}{\partial R} + \dot{T}\frac{\partial\gamma}{\partial T}\Big)\frac{\partial\mathcal{L}}{\partial\dot{T}} = 0.
$$

and by putting the coefficients of $\{\dot{a}^2, \dot{R}^2, \dot{T}^2, \dot{a}\dot{R}, \dot{a}\dot{T}, \dot{T}\dot{R}\}$ and constant terms, we obtain the following system of partial differential equations for $\vec{X}$ components $\{\alpha, \beta, \gamma\}$:

$$
\dot{a}^2: \quad \alpha f_R + \beta a f_{RR} + \gamma a f_{RT} + 2a\alpha_a f_R + a^2 f_{RR}\beta_a + \gamma_a a^2 f_{RT} = 0, \tag{75}
$$

$$
f_{RR}\alpha_R = 0, \quad f_{RT}\alpha_T = 0, \quad \alpha_a f_{RR} = 0 \tag{76}
$$

$$
\dot{a}\dot{R}: \quad a f_{RR}\beta + a\gamma f_{TTT} + a f_{RR}\alpha_a + a\beta_R f_{RR} \tag{77}
$$
$$
+a\gamma R f_{RT} + 2\alpha_R f_R + 2 f_{RR}\alpha = 0,
$$

$$
\dot{a}\dot{T}: \quad a f_{RRR}\beta + a f_{RRT}\gamma + (2\,\alpha + a\alpha_a + a\beta_R)\,f_{RR} + a\gamma_R f_{RT} + 2\alpha_R f_R = 0 \tag{78}
$$

$$
\dot{R}\dot{T}: \alpha_R f_{RT} + \alpha_T f_{RR} = 0 \tag{79}
$$

$$
\Big(\gamma R + \beta\,(T - 3\,p\,(a) + \rho\,(a))\Big)a\,f_{RT} \tag{80}
$$

$$
+\gamma a\Big(T - 3\,p\,(a) + \rho\,(a)\Big)f_{TT} + \beta a R f_{RR}
$$

$$
+3\Big((T - 3p(a) + \rho(a) + \frac{1}{3}a\rho'(a) - ap'(a)\Big)f_T - f + \frac{1}{3}ap'(a) + p(a) + R f_R\alpha = 0.
$$

It is a hard job to find all possible solutions of this nonlinear system of first order coupled partial differential equations (PDEs). Inspired directly from the case of general relativity we'll limit ourselves to the following simple cases:

- The case of Einstein gravity with matter components $f(R,T) = R + 2\Lambda + g(T)$: because any theory of modified gravity must be reduced to Einstein-Hilbert action at low curvature regime, we are interested to study a solution of $f(R,T)$ in which the Einstein-Hilbert term is dominated as the leading term of theory. If we substitute $f(R,T) = R + 2\Lambda + g(T)$, we obtain the

following exact solution:

$$\alpha = \frac{C_1}{\sqrt{a}}, \quad \beta = \gamma = 0 \tag{81}$$

$$g(T) = C_2 e^{-3C_1 \int \frac{dT}{\gamma(T)}} \tag{82}$$

$$p(a) = 2\Lambda + \frac{C_1}{a^3}. \tag{83}$$

We mention here that this is an special class of solutions founded before in literature [61] (for the case of purely $f(R,T) = F(R)$ see [61]). The associated Noether charge reads:

$$- 12 C_1 \dot{a} \sqrt{a} = \Sigma_0. \tag{84}$$

The corresponding scale factor is obtained as follows:

$$a = \left(\frac{3A}{2}\right)^{2/3} \Big[1 - \frac{1}{3} H_0 t \Big]^{2/3}, \quad H_0 = \frac{3\Sigma_0}{8C_1}. \tag{85}$$

This solution can be written in terms of q-exponential family [62] :

$$a(t) = a_0 e_{2/3}(H_0 t), \quad a_0 = \left(\frac{3A}{2}\right)^{2/3}. \tag{86}$$

The pressure term deserves more investigations. The first term implies on the existence of the "background pressure" $p_0 = 2\Lambda$, which it can be realized by expectation vacuum energy of some quantum fields. The second term is a dark matter term, if we identify $c_1 = \rho_{m0}$, as the dark matter density at present era, $t = 0, a_0 = 1$. In the case of perfect fluid with equation of state $w = \frac{p}{\rho}$, we have $\rho \sim a^{-3(1+w)}$. By a power-law expansion, $a(t) \sim t^p$, to be accelerated Universe we must have $p > 1$, in our case the solution is not accelerating solution.

- Case with $f(R,T) = h(R) + k(T)$: if we would like to pass to modified gravity, we should try to solve systems of equations by assuming that $R \rightarrow h(R)$, and by taking into the account the matter sector $k(T)$. This simple assumption gives us the following exact solutions for the model (19) under Noether symmetry approach:

$$\alpha = 0, \quad \beta = \frac{F_1(R)}{a}, \quad \gamma = \frac{R}{a} \Delta(T), \tag{87}$$

$$h(R) = C_1 \int_0^R (R - \tau) F_1(\tau) d\tau, \quad k(T) = C_1 \int \frac{dT}{\Delta(T)} + C_2. \tag{88}$$

here $\{\Delta(T), F_1(R)\}$ stands for a pair of arbitrary functions. The conserved Noether charge associated with this model reads as follows:

$$\sigma_0 = -6 C_1 a \dot{a} F_1(R)^2. \tag{89}$$

Due to the leakage of more information of the form of $F_1(x)$, we cannot integrate it. But it is possible to solve it for Starobinsky inflationary model

$F(R) = R + \frac{R^2}{6M^2}$ solution as the following:

$$t + C_2 = \int_0^{a(t)} \frac{x\,dx}{R(-x^4 - 24j(y)) + 4C_1} \tag{90}$$

here

$$R(-x^4 - 24j(y)) = Root\Big(-x^4 - 24j(y) = 0|_{y=x} \Big), \tag{91}$$

and we define

$$j(y) \equiv \frac{1}{27} \frac{\left(9\,C_1{}^2 y^2 M^2 - C_1 y\sigma_0\right)^{3/2}}{\sigma_0 M^3 C_1{}^2} + \frac{1}{36} \frac{y\sqrt{9\,C_1{}^2 y^2 M^2 - C_1 y\sigma_0}}{M^3 C_1} \tag{92}$$
$$-\frac{1}{648} \frac{\sigma_0 \sqrt{9\,C_1{}^2 y^2 M^2 - C_1 y\sigma_0}}{M^5 C_1{}^2} - \frac{\sqrt{9}\sigma_0^2}{11664 C_1^2 M^6} \ln\Big(\frac{(-\frac{1}{2}C_1\sigma_0 + 9y^2 C_1^2 M^2)}{9C_1 M} \Big)$$
$$+\sqrt{9} M^{-6} C_1^{-2} \sqrt{9C_1^2 y^2 M^2 - C_1 y\sigma_0}) + \frac{y^3 C_1}{\sigma_0}.$$

6. Noether symmetry for mimetic $f(R)$ gravity

In this section we'll see how Noether symmetry gives us useful information about mimetic $f(R)$ theory. Especially we would like to search for the possible forms of potential function $V(\phi)$ in this model. Thanks to the normalization condition $\partial_\mu\phi\partial^\mu\phi = 1$, the potential is an implicit function of t. So, generally speak in the dynamical system Lagrangian is time dependent. Let us to start by the same method as we used in the previous section. We write the following condition of (71) for $f(R)$ mimetic model:

$$\alpha\frac{\partial\mathcal{L}}{\partial a} + \beta\frac{\partial\mathcal{L}}{\partial R} + \gamma\frac{\partial\mathcal{L}}{\partial\phi} + \Big(\dot{a}\frac{\partial\alpha}{\partial a} + \dot{R}\frac{\partial\alpha}{\partial R} + \dot{\phi}\frac{\partial\alpha}{\partial T}\Big)\frac{\partial\mathcal{L}}{\partial\dot{a}} \tag{93}$$
$$+\Big(\dot{a}\frac{\partial\beta}{\partial a} + \dot{R}\frac{\partial\beta}{\partial R} + \dot{\phi}\frac{\partial\beta}{\partial\phi}\Big)\frac{\partial\mathcal{L}}{\partial\dot{R}} + \Big(\dot{a}\frac{\partial\gamma}{\partial a} + \dot{R}\frac{\partial\gamma}{\partial R} + \dot{\phi}\frac{\partial\gamma}{\partial\phi}\Big)\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0.$$

where for simplicity we assume that $\lambda = \lambda(t), p = p(a)$. Using (33), we can write the following system of differential equations , linear in $\{\alpha, \beta, \gamma\}$:

$$-3\alpha R f_R - \beta a R f_{RR} + 3\alpha f - 3\alpha\lambda - 3\alpha V(\phi) - \gamma a V_\phi - \alpha a p'(a) - 3\alpha p = 0 \tag{94}$$
$$3\alpha + 2a\gamma_\phi = 0 \tag{95}$$
$$2\alpha_a a f_R + \beta a f_{RR} + a^2 \beta_a f_{RR} + \alpha f_R = 0 \tag{96}$$
$$\alpha_R f_{RR} = 0 \tag{97}$$
$$\lambda a^2 \gamma_a - 6\alpha_\phi f_R - 3a\beta_\phi f_{RR} = 0 \tag{98}$$
$$2a\alpha_R f_R + 2a\alpha f_{RR} + \beta a^2 f_{RRR} + a^2 \alpha_a f_{RR} + a^2 \beta_R f_{RR} = 0 \tag{99}$$
$$\lambda a\gamma_R - 3\alpha_\phi f_{RR} = 0. \tag{100}$$

which are obtained setting to zero the coefficients of the different terms $\{\dot{a}^2, \dot{R}^2, \dot{\phi}^2, \dot{a}\dot{R}, \dot{a}\dot{\phi}, \dot{\phi}\dot{R}\}$. We can distinguish some possible cases:

- Case of GR: the possible non trivial solution for $f(R) = R + 2\Lambda$ read as the following:

$$V(\phi) = C_4 \tag{101}$$

$$\alpha(a, R, \phi) = \frac{1}{3} a^{1+c_1} C_1 \sqrt{c_1 \lambda} \left(C_3 \sin\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) - C_2 \cos\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) \right), \tag{102}$$

$$\beta(a, R, \phi) = -\frac{(2c_1 + 3) C_1 a^{c_1} \sqrt{c_1 \lambda}}{3R} \left(C_3 \sin\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) - C_2 \cos\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) \right) \tag{103}$$

$$\gamma(a, R, \phi) = C_1 a^{c_1} \left(C_2 \sin\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) + C_3 \cos\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) \right), \tag{104}$$

$$p(a) = 2\Lambda - \lambda - C_4 + a^3 C_5. \tag{105}$$

This solution corresponds to a constant potential form. The Noether conserved charge reads as the following:

$$Q = -4\dot{a}\, a^{2+c_1} C_1 \sqrt{c_1 \lambda} \left(C_3 \sin\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) - C_2 \cos\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) \right) \tag{106}$$
$$+ 2\lambda C_1 a^{c_1 + 3} \left(C_2 \sin\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) + C_3 \cos\left(\frac{\sqrt{c_1 \lambda}}{2}\phi\right) \right)$$

Because $\phi = t$ the equation is integrable to give us $a(t)$. For $Q = 0$, the exact solution existed for the scale factor:

$$a(t) = a_0 \left(C_3 \sin(\omega t) - C_2 \cos(\omega t) \right)^n, \quad n = \frac{1}{c_1}, \quad \omega = \frac{\sqrt{c_1 \lambda}}{2}. \tag{107}$$

It defines an oscillatory solution with Type IV future singularity [63]. If $C_2 = 0, \omega \to i\omega, n = \frac{3}{2}$, It is identified in the late-time ΛCDM era.

Following [64], we can classify the future singularities as follow:

- TypeI: ("BigRip"): $t \to t_s, a \to \infty, \rho \to \infty$ and $|p| \to \infty$.
- TypeII: ("sudden"): $t \to t_s, a \to a_s, \rho \to \rho_s$ and $|p| \to \infty$.
- Type III : $t \to t_s, a \to a_s, \rho \to \infty$ and $|p| \to \infty$
- Type IV : $t \to t_s, a \to a_s, \rho \to 0$ and $|p| \to 0$ and higher derivatives of H diverge.
  Here t_s, a_s and ρ_s are constants with $a_s \neq 0$.

For our scale factor (107) the Hubble parameter and first and second derivatives of H read as the following:

$$H(t) = \frac{\dot{a}}{a} = n\omega \left[\frac{C_3 + C_2 \tan(\omega t)}{C_3 \tan(\omega t) - C_2} \right] \tag{108}$$

$$\dot{H} = -n\omega^2 (C_2^2 + C_3^2) \sec^2(\omega t)(C_3 \tan(\omega t) - C_2)^{-2} \tag{109}$$

$$\ddot{H} = \frac{2n\omega^3 (C_2{}^2 + C_3{}^2)(\sin(\omega t)C_2 + C_3 \cos(\omega t))}{C_3^3 \sin^3(\omega t) - C_2^3 \cos^3(\omega t) - 3C_3 C_2 \left(C_3 \sin(\omega t) - C_2 \cos(\omega t) \right)} \tag{110}$$

There is solution(s) for $H(t_s) = \infty$ where

$$t_s^n = n\pi \pm \arctan\left[\frac{C_{2,3}}{C_2^2 + C_3^2}\right], \quad n \in \mathcal{Z}. \tag{111}$$

Where

$$\lim_{t \to t_s} H^{(n \geq 2)} \to \infty. \tag{112}$$

So, our solution represents Type IV future singularities. It is remarkable that scale factor remains finite at $t \to t_s$. Also, because we must have

$$\lim_{t \to t_s} p(a) = \lim_{a \to a_s} \left(2\Lambda - \lambda - C_4 + a^3 C_5\right) = 0 \tag{113}$$

we obtain:

$$2\Lambda - \lambda - C_4 + a_s{}^3 C_5 = 0 \tag{114}$$

the cosmological constant term is "calibrated" as the following:

$$\Lambda = \frac{1}{2}(\lambda + C_4 - a_s{}^3 C_5). \tag{115}$$

- Modified gravity : if we put $f(R) = R + 2\Lambda + h(R)$ a very careful analysis of the system of equations gives us the following solutions:

6.1. *Solution with quadratic potential form: Hybrid inflation model*

The system of equtions has the following exact solutions for a set of functions:

$$V(\phi) = C_5 + C_6 \left(\phi + \frac{C_2}{C_1}\right)^2, \tag{116}$$

$$\alpha(a, \phi) = -\frac{2}{3} a C_1, \tag{117}$$

$$\beta(a, R, \phi) = 2\frac{C_1}{R}, \tag{118}$$

$$\gamma(a, R, \phi) = C_1 \phi + C_2, \tag{119}$$

$$h(R) = C_3 R + C_4, \tag{120}$$

$$p(a) = -C_5 + 2\Lambda + C_4 - \lambda + a^3 C_7. \tag{121}$$

The quadratic potential is for a massive scalar. This type of potential used as an inflationary model, so called as a Hybrid inflation model. More precisely, if we set $C_2 = 0, C_5 = V_0, C_6 = \frac{m^2}{2}$, the potential in this regime is written as:

$$V(\phi) = V_0 + \frac{m^2}{2}\phi^2. \tag{122}$$

This case is a vacuum energy, drive inflation, and is distinct from the other classes, like large-field or small-field models [65]. Conserved charge is written as the following:

$$\Sigma^* = 8C_1C_3a^2\dot{a} + 2\lambda a^3(C_1t + C_2). \tag{123}$$

Specially by integrating this equation for $\Sigma^* = 0$ we obtain:

$$a(t) = a_0\mathrm{e}^{-\frac{\lambda\, t(C_1t+2\,C_2)}{8C_1C_3}}. \tag{124}$$

This model is described as the bouncing universe [66,67] $a(t) \sim e^{\alpha t^2}$. To show this equivalence, we rewrite it in the following form:

$$a(t) = \tilde{a}_0 e^{-\frac{\lambda C_2^2}{8C_3C_1^2}t^2}, \quad \tilde{a}_0 \equiv a_0e^{\frac{\lambda C_2^2}{C_3C_1^2}}. \tag{125}$$

Hubble parameter and deceleration parameter read as the following:

$$H = -\frac{\lambda C_2^2}{4C_3C_1^2}t, \quad q = -(1 + \frac{\dot{H}}{H^2}) = -(1 - \frac{4C_3C_1^2}{\lambda C_2^2t^2}). \tag{126}$$

To have acceleration expansion we should have:

$$-\frac{\lambda C_2^2}{4C_3C_1^2}(1 - \frac{\lambda C_2^2t^2}{4C_3C_1^2}) > 0. \tag{127}$$

We obtain the following cases:

- If $\frac{\lambda}{C_3} > 0$ then we have acceleration for time intervals $t \in (-\infty, |\frac{4C_3}{\lambda}|^{1/2}\frac{C_1}{C_2})$.
- If $\frac{\lambda}{C_3} < 0$ then we have acceleration for time intervals $t \in (|\frac{4C_3}{\lambda}|^{1/2}\frac{C_1}{C_2}, \infty)$.

In the above discussions, we only consider the case that the universe is expanding.

6.2. *Solution with exponential inflationary models*

A possible solution is the following hyperbolic model:

$$V(\phi) = C_6 + 2C_7\cosh\frac{2(\phi + C_2)}{C_1} \tag{128}$$

$$\alpha(a, \phi) = \frac{2}{3C_1}\,ia^{1+C_4}\sqrt{C_3}\sinh\frac{2(\phi + C_2)}{C_1}, \tag{129}$$

$$\gamma(a, R, \phi) = -ia^{C_4}\sqrt{C_3}\cosh\frac{2(\phi + C_2)}{C_1}, \tag{130}$$

$$\beta(a, R, \phi) = -2i\frac{\sqrt{C_3}\,(2\,C_4 + 3)\,a^{C_4}}{3C_1R}\sinh\frac{2(\phi + C_2)}{C_1}, \tag{131}$$

$$h(R) = -(\frac{1}{4}\lambda\,C_4\,C_1{}^2 + 1)R + C_5, \tag{132}$$

$$p(a) = -(\lambda + C_6 + 6\Lambda + 6C_7 + 3C_5) + C_8a^3 \tag{133}$$

Because all functions must be real valued, so it is adequate to write $C_3 = -M^2$, where M is a free parameter. The associated conserved charge reads:

$$Q_0 = a^{2+C_4}\left[\dot{a}\frac{2(4 - C_4 C_1^2)M}{C_1}\sinh\frac{2(t + C_2)}{C_1} + 2M\lambda a\cosh\frac{2(t + C_2)}{C_1}\right] \quad (134)$$

It is completely integrable,specially when $Q_0 = 0$, we obtain:

$$a(t) = C_1\left(\sinh\left(2\frac{t + C_2}{C_1}\right)\right)^n, \quad n = \frac{\lambda C_1^2}{2(-4 + C_4 C_1^2)}. \quad (135)$$

In case of $n = \frac{3}{2}$ it coincides with the late-time ΛCDM era.

7. Conclusions

Motivated by recent observational data, indicates that we live in an accelerating universe, several forms of modified gravities have been proposed to resolve and explain this physical phenomena. One of the most popular and physically acceptable candidates is $f(R)$ gravity and its extensions. In our work we established Noether symmetry issue for two types of $f(R)$ theories: a type of non-minimally coupled model is called as $f(R,T)$ and mimetic $f(R)$. We started by reviewing the basic physical foundations of these theories. In $f(R,T)$ model we have been written point-like Lagrangian for flat FLRW metric. We studied equations of motion and Noether symmetry form for it. Two important classes of solution for $f(R,T)$ were found. In the first class, we show that the generalized q-exponential scale factor is an exact solution which it mimics the background with the background pressure. Other solutions were found as general family of additive models, with an exact solution for scale factor in terms of elementary functions. This cosmological solution was obtained by considering the Starobinsky model $f(R) = R + \alpha R^2$. In mimetic $f(R)$ theory, we have been considered Noether symmetries. We observed that there are two classes of solutions: the first is equivalent to the GR with dark matter and the calibrated cosmological constant. This case mimics a type of cosmological solutions with type IV future singularities, where higher derivatives of H diverge. Another case is modified gravity with two specified forms of potential functions: hybrid inflationary model,in which the scale factor evolves in the bouncing scenario. The second family is exponential form, in this case scale factor mimics the form of ΛCDM model perfectly. So, all cosmological models including ΛCDM,bouncing and oscillatory solutions with future singularities are described perfectly by Noether symmetrized $f(R,T)$ and mimetic $f(R)$ theories. We conclude that Noether symmetry is able to provide a very excellent way to study cosmological implications of extended $f(R)$ theories.

Acknowledgments

References

[1] A. G. Riess et al.,Observational evidence from supernovae for an accelerating universe and a cosmological constant , Astron. J. **116**, 1009 (1998).

[2] S. Perlmutter et al.,Measurements of Omega and Lambda from 42 high redshift supernovae - Supernova Cosmology Project, Astrophys. J. **517**, 565 (1999).

[3] P. de Bernardis et al.,A Flat universe from high resolution maps of the cosmic microwave background radiation, Nature **404**, 955 (2000).

[4] S. Hanany et al.,MAXIMA-1: A Measurement of the cosmic microwave background anisotropy on angular scales of 10 arcminutes to 5 degrees , Astrophys. J. **545**, L5 (2000).

[5] S. 'i. Nojiri and S. D. Odintsov, Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models, Phys. Rept. **505**, 59 (2011) [arXiv:1011.0544 [gr-qc]].

[6] K. Bamba, S. 'i. Nojiri and S. D. Odintsov, Modified gravity: walk through accelerating cosmology, arXiv:1302.4831 [gr-qc].

[7] K. Bamba and S. D. Odintsov, Universe acceleration in modified gravities: $F(R)$ and $F(T)$ cases, arXiv:1402.7114 [hep-th].

[8] S. 'i. Nojiri and S. D. Odintsov, Modified gravity as realistic candidate for dark energy, inflation and dark matter, AIP Conf. Proc. **1115**, 212 (2009) [arXiv:0810.1557 [hep-th]].

[9] S. Capozziello and M. De Laurentis, Extended Theories of Gravity, Phys. Rept. **509**, 167 (2011) [arXiv:1108.6266 [gr-qc]].

[10] S. Capozziello, L. Consiglio, M. De Laurentis, G. De Rosa and C. Di Donato, The missing matter problem: from the dark matter search to alternative hypotheses, arXiv:1110.5026 [astro-ph.CO].

[11] S. Capozziello and A. Stabile, The Weak Field Limit of Fourth Order Gravity, In *Frignanni, Vincent R. (ed.): Classical and Quantum Gravity: Theory and Applications* Chapter 2 (Nova Press) [arXiv:1009.3441 [gr-qc]].

[12] S. Capozziello and M. De Laurentis, A Review about Invariance Induced Gravity: Gravity and Spin from Local Conformal-Affine Symmetry, Found. Phys. **40** (2010) 867 [arXiv:0910.2881 [hep-th]].

[13] S. Capozziello and M. Francaviglia,Extended Theories of Gravity and their Cosmological and Astrophysical Applications, Gen. Rel. Grav. **40** (2008) 357 [arXiv:0706.1146 [astro-ph]].

[14] S. Nojiri and S. D. Odintsov, Introduction to modified gravity and gravitational alternative for dark energy, eConf C **0602061**, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)] [hep-th/0601213].

[15] H. A. Buchdahl,Non-linear Lagrangians and cosmological theory , Mon. Not. Roy. Astron. Soc. 150, 1 (1970).

[16] F. S. N. Lobo,Beyond Einstein's General Relativity, arXiv:1412.0867 [gr-qc]

[17] T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, $f(R,T)$ gravity, Phys. Rev. D **84**, 024020 (2011) [arXiv:1104.2669 [gr-qc]].

[18] M. Jamil, D. Momeni, M. Raza and R. Myrzakulov, Reconstruction of some cosmological models in f(R,T) gravity, Eur. Phys. J. C **72**, 1999 (2012) [arXiv:1107.5807 [physics.gen-ph]].

[19] F. G. Alvarenga, A. de la Cruz-Dombriz, M. J. S. Houndjo, M. E. Rodrigues and D. Sez-Gmez, Dynamics of scalar perturbations in f(R,T) gravity, Phys. Rev. D **87**, no. 10, 103526 (2013) [arXiv:1302.1866 [gr-qc]].

[20] S. D. Odintsov and D. Sez-Gmez, $f(R,T,R_{\mu\nu}T^{\mu\nu})$ gravity phenomenology and ΛCDM universe, Phys. Lett. B **725**, 437 (2013) [arXiv:1304.5411 [gr-qc]].

20   *D. Momeni et al.*

[21] M. Jamil, D. Momeni and R. Myrzakulov, Violation of First Law of Thermodynamics in f(R,T) Gravity, Chin. Phys. Lett. **29**, 109801 (2012) [arXiv:1209.2916 [physics.gen-ph]].

[22] H. Shabani and M. Farhoudi, f(R,T) Cosmological Models in Phase Space, Phys. Rev. D **88**, 044048 (2013) [arXiv:1306.3164 [gr-qc]].

[23] B. Mirza and F. Oboudiat, $f(R,T)$ and future singularities, arXiv:1412.6640 [gr-qc].

[24] T. Harko, Thermodynamic interpretation of the generalized gravity models with geometry - matter coupling, Phys. Rev. D **90**, no. 4, 044067 (2014) [arXiv:1408.3465 [gr-qc]].

[25] A. F. Santos, Gödel solution in $f(R,T)$ gravity, Mod. Phys. Lett. A **28**, 1350141 (2013) [arXiv:1308.3503 [gr-qc]].

[26] P. H. R. S. Moraes, Cosmology from induced matter model applied to 5D $f(R,T)$ theory, Astrophys. Space Sci. **352**, 273 (2014).

[27] M. J. S. Houndjo, D. Momeni and R. Myrzakulov, Cylindrical Solutions in Modified f(T) Gravity, Int. J. Mod. Phys. D **21**, 1250093 (2012) [arXiv:1206.3938 [physics.gen-ph]].

[28] E. H. Baffou, A. V. Kpadonou, M. E. Rodrigues, M. J. S. Houndjo and J. Tossa, Cosmological viable f(R,T) dark energy model: dynamics and stability, Astrophys. Space Sci. **355**, 2197 (2014) [arXiv:1312.7311 [gr-qc]].

[29] H. Shabani and M. Farhoudi, Cosmological and Solar System Consequences of f(R,T) Gravity Models, Phys. Rev. D **90**, 044031 (2014) [arXiv:1407.6187 [gr-qc]].

[30] X. M. Deng and Y. Xie, Solar Systems Bounds on the Extra Acceleration of f(R, T) Gravity Revisited, Int. J. Theor. Phys. (2014).

[31] C. J. Ferst and A. F. Santos, Gödel-type solution in $f(R,T)$ modified gravity, arXiv:1411.1002 [gr-qc].

[32] V. Singh and C. P. Singh, Friedmann cosmology with particle creation in modified $f(R,T)$ gravity, arXiv:1408.0633 [gr-qc].

[33] C. P. Singh and P. Kumar, Friedmann model with viscous cosmology in modified $f(R,T)$ gravity theory, Eur. Phys. J. C **74**, no. 10, 3070 (2014) [arXiv:1406.4258 [gr-qc]].

[34] A. H. Chamseddine and V. Mukhanov, Mimetic Dark Matter, JHEP **1311**, 135 (2013) `arXiv:1308.5410`.

[35] A. Golovnev, On the recently proposed Mimetic Dark Matter, Phys. Lett. B **728**, 39 (2014) `arXiv:1310.2790`.

[36] A. H. Chamseddine, V. Mukhanov and A. Vikman, Cosmology with Mimetic Matter, JCAP **1406**, 017 (2014) `arXiv:1403.3961`.

[37] M. Chaichian, J. Kluson, M. Oksanen and A. Tureanu, Mimetic dark matter, ghost instability and a mimetic tensor-vector-scalar gravity, JHEP **1412**, 102 (2014) [arXiv:1404.4008 [hep-th]].

[38] D. Momeni, P. H. R. S. Moraes, H. Gholizade and R. Myrzakulov, Mimetic Compact Stars, arXiv:1505.05113 [gr-qc].

[39] O. Malaeb, Hamiltonian Formulation of Mimetic Gravity, arXiv:1404.4195.

[40] N. Deruelle and J. Rua, Disformal Transformations, Veiled General Relativity and Mimetic Gravity, JCAP **1409**, 002 (2014), ,arXiv:1407.0825.

[41] J. Matsumoto, S. D. Odintsov and S. V. Sushkov, Cosmological perturbations in mimetic matter model, arXiv:1501.02149 [gr-qc].

[42] D. Momeni, A. Altaibayeva and R. Myrzakulov, New Modified Mimetic Gravity, arXiv:1407.5662.

[43] S. Nojiri and S. D. Odintsov, Mimetic $F(R)$ gravity: inflation, dark energy and bounce, arXiv:1408.3561.

[44] G. Leon and E. N. Saridakis, Dynamical behavior in mimetic F(R) gravity, arXiv:1501.00488 [gr-qc].

[45] M. Mehrpooya and D. Momeni, Spherically symmetric massive scalar fields in general relativity, Int. J. Mod. Phys. A **25**, 1429 (2010) [arXiv:0903.1185 [gr-qc]].

[46] S. Capozziello, M. De Laurentis and S. D. Odintsov, Hamiltonian dynamics and Noether symmetries in Extended Gravity Cosmology, Eur. Phys. J. C **72**, 2068 (2012) [arXiv:1206.4842 [gr-qc]].

[47] R.de Ritis, G. Marmo, G. Platania, C. Rubano, P. Scudellaro and C. Stornaiolo ,New approach to find exact solutions for cosmological models with a scalar field , *Phys. Rev. D* **42** 1091 (1990).

[48] S. Capozziello and R. de Ritis, Noether's symmetries and exact solutions in flat nonminimally coupled cosmological models, *Class. Quant. Grav.* **11** 107 (1994).

[49] S. Capozziello and R. de Ritis, *Phys. Lett. A* **195** 48 (1994).

[50] S. Capozziello, R. de Ritis,] Relation between the potential and nonminimal coupling in inflationary cosmology , Phys. Lett. A **177**, 1 (1993)

[51] S. Capozziello, Lambiase,Selection rules in minisuperspace quantum cosmology, Gen. Relativ. Gravit. **32**, 673 (2000)

[52] S. Capozziello, M. De Laurentis and S. D. Odintsov, Noether Symmetry Approach in Gauss-Bonnet Cosmology, arXiv:1406.5652 [gr-qc].

[53] A. Paliathanasis, M. Tsamparlis, S. Basilakos, S. Capozziello,Scalar-Tensor Gravity Cosmology: Noether symmetries and analytical solutions, arXiv:1403.0332 [astro-ph.CO] (2014).

[54] M. Jamil, D. Momeni and R. Myrzakulov, Energy conditions in generalized teleparallel gravity models, Gen. Rel. Grav. **45**, 263 (2013) [arXiv:1211.3740 [physics.gen-ph]].

[55] Y. Kucukakca, Scalar tensor teleparallel dark gravity via Noether symmetry, Eur. Phys. J. C **73**, 2327 (2013) [arXiv:1404.7315 [gr-qc]].

[56] A. Aslam, M. Jamil, D. Momeni, R. Myrzakulov, M. A. Rashid and M. Raza, Noether gauge symmetry approach in quintom cosmology, Astrophys. Space Sci. **348**, 533 (2013) [arXiv:1308.2221 [astro-ph.CO]].

[57] A. Aslam, M. Jamil, D. Momeni and R. Myrzakulov, Noether Gauge Symmetry of Modified Teleparallel Gravity Minimally Coupled with a Canonical Scalar Field, Can. J. Phys. **91**, 93 (2013) [arXiv:1212.6022 [astro-ph.CO]].

[58] M. Jamil, D. Momeni and R. Myrzakulov, Noether symmetry of F(T) cosmology with quintessence and phantom scalar fields, Eur. Phys. J. C **72**, 2137 (2012) [arXiv:1210.0001 [physics.gen-ph]].

[59] M. Jamil, S. Ali, D. Momeni and R. Myrzakulov, Bianchi Type I Cosmology in Generalized Saez-Ballester Theory via Noether Gauge Symmetry, Eur. Phys. J. C **72**, 1998 (2012) [arXiv:1201.0895 [physics.gen-ph]].

[60] M. Jamil, F. M. Mahomed and D. Momeni, Noether Symmetry Approach in f(R) Tachyon Model, Phys. Lett. B **702** (2011) 315 [arXiv:1105.2610 [physics.gen-ph]].

[61] S. Capozziello and A. De Felice, f(R) cosmology by Noether's symmetry, JCAP **0808**, 016 (2008) [arXiv:0804.2163 [gr-qc]].

[62] M. R. Setare, D. Momeni, V. Kamali and R. Myrzakulov, Inflation driven by q-de Sitter in light of Planck 2013 and BICEP2 results, arXiv:1409.3200 [physics.gen-ph].

[63] S. Nojiri and S. D. Odintsov, Inhomogeneous equation of state of the universe: Phantom era, future singularity and crossing the phantom barrier, Phys. Rev. D **72**, 023003 (2005) [hep-th/0505215].

[64] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Properties of singularities in (phantom) dark energy universe, Phys. Rev. D **71**, 063004 (2005) [hep-th/0501025].

[65] J.Yokoyama,Inflation: 1980201X, Prog. Theor. Exp. Phys. 2014, 06B103,DOI:

22 *D. Momeni et al.*

10.1093/ptep/ptu081.

[66] R. H. Brandenberger,The Matter Bounce Alternative to Inflationary Cosmology, arXiv:1206.4196 [astro-ph.CO].

[67] Y. F. Cai, Exploring Bouncing Cosmologies with Cosmological Surveys, Sci. China Phys. Mech. Astron. **57**, 1414 (2014) [arXiv:1405.1369 [hep-th]].